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Coadjoint orbits of extensions of $Diff^{+}(S^1)$ by modules of tensor densities

Christophe Kapoudjian

Institut Girard Desargues, UPRES-A 5028 du CNRS, Université Claude Bernard Lyon-I, 43, Boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex, France

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Abstract

We investigate extensions of the diffeomorphism group of the circle by modules of λ-densities — as well as their Lie algebras — from the point of view of the coadjoint representation. The orbits of a moment μ are classified according to the zeroes of the functions involved in μ, in the spirit of the classification of the Bott–Virasoro orbits.

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0. Introduction

The classification of coadjoint orbits of the Bott–Virasoro group — investigated by Kirillov, Lazutkin and Pankratova, Segal, Witten, and completed by Guieu (cf. [3,4]) — has proved to be of great interest in the understanding of the geometry of some related structures, such as the space of Hill operators and the space of projective structures on the circle.

A natural generalization of the Bott–Virasoro group (resp. the Virasoro algebra) is given by extensions of the group $Diff^{+}(S^1)$ of orientation-preserving diffeomorphisms of the circle (resp. extensions of the Lie algebra $Vect(S^1)$ of vector fields on the circle) by modules $\mathcal{F}_\lambda$ of λ-densities on the circle. The problem of classifying such extensions is equivalent to that of the calculation of the cohomology groups $H^2_{diff}(Diff^{+}(S^1); \mathcal{F}_\lambda)$ and $H^2(Vect(S^1); \mathcal{F}_\lambda)$. The latter was determined in [10] (in [1] the formal case was solved), but the associated

E-mail address: ckapoudj@desargues.univ-lyon1.fr (C. Kapoudjian).

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cocycles were found in [9], as well as the calculation of $H^2_{diff}(Diff^+(S^1); \mathcal{F}_\lambda)$, with explicit formulas for the group cocycles (Section 1). Mysteriously, these second cohomology groups happen to be non-trivial only for $\lambda \in \{0, 1, 2, 5, 7\}$.

The present work proposes to study the coadjoint orbits of the corresponding extensions. New phenomena in the classification of orbits appear when λ exceeds the critical value $\lambda = 1$ (Section 4): if μ is an element of the dual of the Lie algebra, the multiplicities of the zeroes of the functions involved in μ play an essential role in the classification. In the case $\lambda = 2$ (detailed in Section 5), the Lie algebra $Vect(S^1) \rhd \mathcal{F}_2$ happens to be isomorphic to its regular dual space: this leads to a classification slightly different from cases $\lambda = 5$ and 7. As for the case $\lambda = 0$ (Section 3), it is equivalent to that of a Virasoro-type central extension of the semi-direct group product $Diff^+(S^1) \rhd C^\infty(S^1, \mathbb{R})$, and is developed in Section 7.

1. Cohomological results

Denote by $Diff^+(S^1)$ the Lie–Fréchet group of orientation-preserving diffeomorphisms of the circle. It will be convenient to work with its universal covering, the group $Diff_{\mathbb{Z}}(\mathbb{R})$ of $\mathbb{Z}$-equivariant diffeomorphisms of the real line:

$$Diff_{\mathbb{Z}}(\mathbb{R}) = \{\tilde{\phi} \in Diff(\mathbb{R}) | \tilde{\phi}(x+1) = \tilde{\phi}(x) + 1 \quad \forall x \in \mathbb{R}\}.$$

Each $\tilde{\phi} \in Diff_{\mathbb{Z}}(\mathbb{R})$ induces a diffeomorphism $\phi \in Diff^+(S^1)$ defined by $\phi(\mathrm{e}^{2\mathrm{i}\pi x}) = \mathrm{e}^{2\mathrm{i}\pi\tilde{\phi}(x)}$.

Now let $\mathcal{F}_\lambda$ be the space of smooth tensor densities on S^1 of degree λ, with $\lambda \in \mathbb{R}$:

$$\mathcal{F}_\lambda = \{a(x)(\mathrm{d}x)^\lambda | a \in C^\infty(S^1, \mathbb{R})\}.$$

($C^\infty(S^1, \mathbb{R})$ is identified with the space of 1-periodic smooth functions of $\mathbb{R}$). Geometrically, $\mathcal{F}_\lambda$ is the space of sections of the fibre bundle $(T^* S^1)^{\otimes^\lambda}$ over S^1. Each orientation-preserving diffeomorphism ϕ on the base induces an automorphism ϕ^* of $\mathcal{F}_\lambda$, so that $\mathcal{F}_\lambda$ naturally inherits a $Diff^+(S^1)$-module structure, the anti-action of $\phi \in Diff^+(S^1)$ on $a \in \mathcal{F}_\lambda$ being defined by

$$\phi^*(a(\mathrm{d}x)^\lambda) = a \circ \tilde{\phi}(\tilde{\phi}')^\lambda(\mathrm{d}x)^\lambda.$$

In the following, we will often write ϕ instead of $\tilde{\phi}$.

Differentiating this action leads to a $Vect(S^1)$-module structure of on $\mathcal{F}_\lambda$, where $Vect(S^1)$ is the Lie–Fréchet algebra of vector fields on the circle: a vector field $f = f\,\mathrm{d}/\mathrm{d}x$ acts on a λ-density $a\,(\mathrm{d}x)^\lambda$ according to the formula:

$$L_f(a(\mathrm{d}x)^\lambda) = (fa' + \lambda f'a)(\mathrm{d}x)^\lambda.$$

In [9], the classification of all non-trivial extensions

$$0 \to \mathcal{F}_\lambda \longrightarrow G_\lambda \longrightarrow Diff^+(S^1) \to 1$$

of the group $Diff^+(S^1)$ by the $Diff^+(S^1)$-module $\mathcal{F}_\lambda$ has been obtained. The arrows are supposed to be smooth, and the fibration over $Diff^+(S^1)$ is topologically trivial (since the fibre is contractible): G_λ is diffeomorphic to $Diff^+(S^1) \times \mathcal{F}_\lambda$. The group structure on G_λ is given by a product of the form

$$(\phi, a)(\psi, b) = (\phi \circ \psi, b + \psi^* a + B_\lambda(\phi, \psi))$$

where B_λ is a smooth two-cocycle with values in the module $\mathcal{F}_\lambda$:

$$B_\lambda(\phi, \psi \circ \chi) + B_\lambda(\psi, \chi) = B_\lambda(\phi \circ \psi, \chi) + \chi^* B_\lambda(\phi, \psi).$$

As is well known, such extensions are classified by the cohomology classes of smooth cocycles, i.e. by the second space of differentiable cohomology of $Diff^+(S^1)$, with coefficients in $\mathcal{F}_\lambda$, denoted $H^2_{diff}(Diff^+(S^1); \mathcal{F}_\lambda)$.

Before giving their result, recall that the following map:

$$\begin{aligned} l &: \phi \mapsto \log(\tilde{\phi}') \\ \mathrm{d}l &: \phi \mapsto \mathrm{d}\log(\tilde{\phi}') = \frac{\tilde{\phi}''}{\tilde{\phi}'}\,\mathrm{d}x \\ S &: \phi \mapsto \left(\frac{\tilde{\phi}'''}{\tilde{\phi}'} - \frac{3}{2}\left(\frac{\tilde{\phi}''}{\tilde{\phi}'}\right)^2 \right)(\mathrm{d}x)^2 \end{aligned}$$

define (differentiable) one-cocycles with values in the the modules $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. $S(\phi)$ is the Schwartzian derivative of ϕ.

Denote by B the $\mathbb{R}$-valued Bott–Thurston cocycle on $Diff^+(S^1)$:

$$B(\phi, \psi) = \int_{S^1} \log(\phi \circ \psi)' \,\mathrm{d}\log \psi',$$

and by B_0 the $\mathcal{F}_0$-valued cocycle on $Diff^+(S^1)$, such that $B_0(\phi, \psi)$ is the constant function equal to $B(\phi, \psi)$.

Theorem 1 [9]. *The cohomology groups $H^2_{diff}(Diff^+(S^1); \mathcal{F}_\lambda)$, where $\lambda = 0, 1, 2, 5, 7$ are one-dimensional, generated by the following non-trivial two-cocycles:*

(1) $B_0(\phi, \psi) = \mathrm{const}(\phi, \psi) = B(\phi, \psi)$,

(2) $B_1(\phi, \psi) = \psi^*(l\phi)\,\mathrm{d}l\psi$,

(3) $B_2(\phi, \psi) = \psi^*(l\phi) S\psi$,

(4) $B_5(\phi, \psi) = \begin{vmatrix} \psi^* S\phi & S\psi \\ (\psi^* S\phi)' & (S\psi)' \end{vmatrix}$,

(5) $B_7(\phi, \psi) = 2\begin{vmatrix} \psi^* S\phi & S\psi \\ (\psi^* S\phi)''' & (S\psi)''' \end{vmatrix} - 9\begin{vmatrix} (\psi^* S\phi)' & (S\psi)' \\ (\psi^* S\phi)'' & (S\psi)'' \end{vmatrix}$

$$- \tfrac{32}{3}(S\psi + S(\phi \circ \psi)) B_5(\phi, \psi)$$

If $\lambda \neq 0, 1, 2, 5, 7$, then $H^2_{diff}(Diff^+(S^1); \mathcal{F}_\lambda) = 0$.

The authors of [9] also classify the non-trivial extensions of the Lie algebra $Vect(S^1)$ by the modules $\mathcal{F}_\lambda$:

$$0 \to \mathcal{F}_\lambda \longrightarrow \mathfrak{g}_\lambda \longrightarrow Vect(S^1) \to 0,$$

the Lie structure on $\mathfrak{g}_\lambda = Vect(S^1) \oplus \mathcal{F}_\lambda$ being given by

$$[(f,a),(g,b)] = ([f,g], L_f b - L_g a + c(a,b)),$$

where c is a two-cocycle with values in $\mathcal{F}_\lambda$.

The dimension of $H^2(Vect(S^1); \mathcal{F}_\lambda)$ was found in [10]. But the following theorem provides cocycles representing the cohomology classes:

Theorem 2 [9]. *The second cohomology group, in the sense of Gelfand–Fuchs, of the Lie algebra $Vect(S^1)$, with coefficients in $\mathcal{F}_\lambda$, is*

$$\begin{aligned} H^2(Vect(S^1); \mathcal{F}_\lambda) &= \mathbb{R}^2 \quad \text{if } \lambda = 0, 1, 2 \\ &= \mathbb{R} \quad \text{if } \lambda = 5, 7 \\ &= 0 \quad \text{if } \lambda \neq 0, 1, 2, 5, 7. \end{aligned}$$

The cohomology groups $H^2(Vect(S^1); \mathcal{F}_\lambda)$, where $\lambda = 0, 1, 2, 5, 7$ are generated by the cohomology classes of the following eight non-trivial two-cocycles:

(1) $\bar{c}_0(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix},$

(2) $c_0(f,g) = \text{const}(f,g) = \omega(f,g),$

(3) $c_1(f,g) = \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} \mathrm{d}x,$

(4) $\bar{c}_1(f,g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} \mathrm{d}x,$

(5) $c_2(f,g) = \begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix} (\mathrm{d}x)^2,$

(6) $\bar{c}_2(f,g) = \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} (\mathrm{d}x)^2,$

(7) $c_5(f,g) = \begin{vmatrix} f''' & g''' \\ f^{(4)} & g^{(4)} \end{vmatrix} (\mathrm{d}x)^5,$

(8) $c_7(f,g) = \left(2\begin{vmatrix} f''' & g''' \\ f^{(6)} & g^{(6)} \end{vmatrix} - 9\begin{vmatrix} f^{(4)} & g^{(4)} \\ f^{(5)} & g^{(5)} \end{vmatrix}\right) (\mathrm{d}x)^7,$

where ω denotes the Gelfand–Fuchs cocycle: $\omega(f,g) = \int_0^1 (f''g' - g''f')\,\mathrm{d}x$.

The algebra cocycles c_0, c_1, c_2, c_5, c_7 correspond to the group cocycles B_0, B_1, B_2, B_5, B_7, whereas the algebra cocycles $\bar{c}_0, \bar{c}_1, \bar{c}_2$ cannot be "integrated" to the group $Diff^+(S^1)$ (they are not $\mathfrak{so}(2,\mathbb{R})$-basic, cf. Van-Est theorem in [1]).

2. Coadjoint actions of $\mathfrak{g}_\lambda = Vect(S^1) \rhd \mathcal{F}_\lambda$ and of its extensions

The regular dual (cf. [6]) of $\mathcal{F}_{1-\lambda}$ is identified with the space of distributions of the form

$$I_{u\,(\mathrm{d}x)^\lambda} : a(x)(\mathrm{d}x)^{1-\lambda} \mapsto \int_{S^1} a(x)u(x)\ \mathrm{d}x,$$

where u belongs to $\mathcal{C}^\infty(S^1, \mathbb{R})$. We thus obtain an isomorphism of $Vect(S^1)$-modules:

$$I : \mathcal{F}_\lambda \longrightarrow (\mathcal{F}_{1-\lambda})^*, \quad u(\mathrm{d}x)^\lambda \mapsto I_{u(\mathrm{d}x)^\lambda},$$

i.e. $I_{L_f(u\,(\mathrm{d}x)^\lambda)} = (L_f)^* I_{u\,(\mathrm{d}x)^\lambda}$ for all $f\,\mathrm{d}/\mathrm{d}x \in Vect(S^1)$ and all $u \in \mathcal{C}^\infty(S^1, \mathbb{R})$, with $(L_f)^* = -{}^{\mathrm{t}}L_f$ in the sense of duality.

In particular, the regular dual of $Vect(S^1) = \mathcal{F}_{-1}$ is $\mathcal{F}_2$, the space of two-densities. We denote by $\mu = (u(\mathrm{d}x)^2, v(\mathrm{d}x)^{1-\lambda})$ any element of the regular dual $(Vect(S^1) \oplus \mathcal{F}_\lambda)^* \cong \mathcal{F}_2 \oplus \mathcal{F}_{1-\lambda}$.

In general, if ℓ is a Lie algebra and ℓ^* its dual space, the coadjoint action of ℓ on ℓ^* is defined by $ad^* = -{}^{\mathrm{t}}ad$, where ${}^{\mathrm{t}}ad$ is the transpose of the adjoint action of ℓ on itself. For all ξ in ℓ, $ad^*(\xi)$ lives in $End(\ell^*)$.

If L is a Lie group integrating the algebra ℓ, the coadjoint action of L on ℓ^* is defined by $Ad^*(\phi) ={}^{\mathrm{t}} Ad(\phi^{-1})$ for all $\phi \in L$, and $Ad^*(\phi)$ lies in $GL(\ell^*)$.

In the orbit program of Kirillov (cf. [7]), we are especially interested in the classification of coadjoint orbits of a Lie group. If μ belongs to ℓ^*, denote by $\mathcal{O}_\mu$ the coadjoint orbit passing through μ:

$$\mathcal{O}_\mu = \{Ad^*(\phi)(\mu) \mid \phi \in L\} \cong L/Stab_\mu,$$

where $Stab_\mu$ is the isotropy group, or stabilizer, of μ.

The tangent space at μ to the group orbit is

$$T_\mu \mathcal{O}_\mu = \{ad^*(\xi)(\mu) \mid \xi \in \ell\} \cong \ell/stab_\mu,$$

where $stab_\mu$ is the isotropy algebra of μ.

The orbit is endowed with a symplectic (presymplectic in the infinite dimensional case) L-invariant form, the Kostant–Kirillov–Souriau form, whose restriction at $T_\mu \mathcal{O}_\mu$ is

$$\omega_\mu(ad^*(\xi)(\mu), ad^*(\eta)(\mu)) = \langle \mu, [\xi, \eta] \rangle.$$

In case L is a group of transformations of a manifold ($L = Diff^+(S^1)$ in our context), and when the isotropy algebra is finite dimensional, the Lie–Palais theorem asserts it is exactly the Lie algebra of the isotropy group. Indeed, apply the following theorem to $G = Stab_\mu$:

Theorem 3 (Palais, cf. [8], I. Theorem 3.1). *Let G be a group of differentiable transformations of a manifold M. Let S be the set of all vector fields ξ which generate global 1-parameter groups $\phi_t = \exp(t\xi)$ of transformations of M such that for all t, $\phi_t \in G$. If the set S generates a finite-dimensional Lie algebra of vector fields on M, then G is a Lie transformation group and S is the Lie algebra of G.*

Consequently, if $stab_\mu$ is finite dimensional, we can recover from $stab_\mu$ the connected component of the identity of $Stab_\mu$, but the set of connected components $\pi_0(Stab_\mu)$ is lost. In the following, we study the stabilizers of moments μ for the extensions of the Lie algebras and groups given by Theorems 1 and 2, trying, in some cases, to know more about $\pi_0(Stab_\mu)$, using a theorem of Guieu obtained in [3] (cf. Section 7).

Let us conclude this section with a remark: The coadjoint action of $Vect(S^1) \oplus \mathcal{F}_\lambda$ on $\mathcal{F}_2 \oplus \mathcal{F}_{1-\lambda}$ restricts to the adjoint action of $Vect(S^1)$ on $\mathcal{F}_{1-\lambda}$:

$$L_{f\,\mathrm{d}/\mathrm{d}x} v(\mathrm{d}x)^{1-\lambda} = (fv' + (1-\lambda)f'v)(\mathrm{d}x)^{1-\lambda}.$$

The differential equation $fv' + (1-\lambda)f'v = 0$, that will appear in determining our isotropy algebras, behaves differently according to the values of λ: First, if $\lambda = 1$, this is no longer a true differential equation. If $\lambda \neq 1$, on each interval $]x_0, y_0[$ on which v does not vanish, $f = Av^{1/(\lambda-1)}$, for some $A \in \mathbb{R}$. So, either $\lambda < 1$ and then, if v has a zero x_0, the only solution corresponds to $A = 0$, i.e. $f \equiv 0$; or $\lambda > 1$, and v may vanish without forcing f to be trivial, provided the multiplicity of the zero x_0 is a multiple of $\lambda - 1$. Thus, in the following classification, the value $\lambda = 1$ appears as a critical value, and the cases $\lambda = 2, 5, 7$ will reveal classifications of orbits depending on the multiplicities of the zeroes of v.

3. The case $\lambda = 0$: extensions of the gauge group of the line bundle over the circle

Consider the tangent bundle of the circle as an $\mathbb{R}$-principal (trivial) bundle. The semi-direct product $Diff^+(S^1) \rhd C^\infty(S^1, \mathbb{R})$ is the preserving-orientation automorphism group (or gauge group) of this bundle.

3.1. Extension by the cocycle $\bar{c}_0$

Proposition 1. *Let $(Vect(S^1) \oplus \mathcal{F}_0, k\bar{c}_0)$ be the extension of $Vect(S^1)$ by $\mathcal{F}_0$ realized by the cocycle $k\bar{c}_0$, with $k \in \mathbb{R}^*$. The coadjoint action of $(Vect(S^1) \oplus \mathcal{F}_0, k\bar{c}_0)$ on its regular dual $\mathcal{F}_2 \oplus \mathcal{F}_1$ is given by*

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)(u(\mathrm{d}x)^2, v\,\mathrm{d}x) = ((fU' + 2f'U + a'v)(\mathrm{d}x)^2, (fv)'\,\mathrm{d}x),$$

where $U = u + kv$.

It follows that the stabilizer of $\mu = (u(\mathrm{d}x)^2, v\ \mathrm{d}x)$ is isomorphic to that of $\hat{\mu} = (U(\mathrm{d}x)^2, v\ \mathrm{d}x)$ under the coadjoint action of the true semi-direct product $\mathfrak{g}_0 = Vect(S^1) \rhd \mathcal{F}_0$. The classification of the stabilizers of $\mu = (u(\mathrm{d}x)^2, v\,\mathrm{d}x)$ organizes in the following way:

1. *If v nowhere vanishes, $stab_\mu = \{(f\,\mathrm{d}/\mathrm{d}x, a) : f = A/v, a = -(A/k)[(u+kv)/v^2] + B,\ (A, B) \in \mathbb{R}^2\}$. It is a two-dimensional abelian Lie algebra.*
2. *If $Z_v := \{x \in S^1 : v(x) = 0\}$ is not empty, then*
   - *either the interior of Z_v, $\overset{\circ}{Z}_v$, is empty, and then*

$$stab_\mu = \left\{\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right) = (0, A),\ A \in \mathbb{R}\right\} \cong \mathbb{R}$$

   is one-dimensional,

- *or $\overset{\circ}{Z}_v$ is not empty, and then*

$$stab_\mu = \left\{\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right) : supp\ f \subset Z_u \cap Z_v \text{ and } supp\ a' \subset Z_v\right\}$$

is infinite dimensional.

Proof. The bracket on $(Vect(S^1) \oplus \mathcal{F}_0, k\bar{c}_0)$ is

$$[(f, a), (g, b)] = ([f, g], fb' - ga' + k(fg' - gf')).$$

It follows that the coadjoint action is

$$\begin{aligned}
&\left\langle ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)(u(\mathrm{d}x)^2, v\,\mathrm{d}x), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right\rangle \\
&= -\left\langle (u(\,\mathrm{d}x)^2, v\,\mathrm{d}x), ad\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)\left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right\rangle \\
&= -\langle (u(\mathrm{d}x)^2, v\,\mathrm{d}x), (fg' - gf'), (fb' - ga' + k(fg' - gf'))\rangle \\
&= -\int_{S^1} u(fg' - gf')\,\mathrm{d}x - \int_{S^1} v(fb' - ga' + k(fg' - gf'))\,\mathrm{d}x \\
&= \int_{S^1} (((u + kv)f)' + (u + kv)f')g\,\mathrm{d}x + \int_{S^1} a'vg\,\mathrm{d}x + \int_{S^1} (vf)'b\,\mathrm{d}x,
\end{aligned}$$

which gives the formula of the coadjoint action.

$$\text{Thus, } \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right) \in stab_\mu \iff \begin{cases} 2f'U + fU' + va' = 0 \\ (vf)' = 0 \end{cases}$$

We refer to Section 7 for the proof of the classification. □

3.2. *Extension by the constant Gelfand–Fuchs cocycle c_0*

Consider the central extension of the semi-direct product $\mathfrak{g}_0 = Vect(S^1) \rhd \mathcal{F}_0$ obtained by the $\mathbb{R}$-valued Gelfand–Fuchs cocycle $k\omega$ (cf. Section 7). It is denoted $\widehat{\mathfrak{g}_0}$. The Lie bracket on $Vect(S^1) \rhd \mathcal{F}_0 \oplus \mathbb{R}$ is

$$[((f, a), \sigma), ((g, b), \tau)] = (([f, g], fb' - ga'), k\omega(f, g)).$$

Proposition 2. *Let $(Vect(S^1) \oplus \mathcal{F}_0, kc_0)$ be the extension of $Vect(S^1)$ by $\mathcal{F}_0$ realized by the cocycle kc_0, with $k \in \mathbb{R}$.*

The coadjoint action of $\left(Vect(S^1) \oplus \mathcal{F}_0, kc_0\right)$ on its regular dual space $\mathcal{F}_2 \oplus \mathcal{F}_1$ is given by

$$\begin{aligned}
&ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)(u(\mathrm{d}x)^2, v\,\mathrm{d}x) \\
&= \left(\left(2k\left(\int_{S^1} v\,\mathrm{d}x\right) f''' + fu' + 2f'u + a'v\right)(\mathrm{d}x)^2, (fv)'\,\mathrm{d}x\right).
\end{aligned}$$

The stabilizer $stab_\mu$ *of* $\mu = (u(\mathrm{d}x)^2, v\,\mathrm{d}x)$ *for the coadjoint action coincides with the stabilizer of the moment* $\mu \oplus \int_{S^1} v\,\mathrm{d}x \in (Vect(S^1) \triangleright \mathcal{F}_0 \oplus \mathbb{R})^* = \widehat{\mathfrak{g}_0}^*$ *for the coadjoint action of* $\mathfrak{g}_0$.

Proof.

$$\left\langle ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)(u(\mathrm{d}x)^2, v\,\mathrm{d}x), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right\rangle$$
$$= \left\langle (fu' + 2f'u + a'v)(\mathrm{d}x)^2, g\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle - k\int_{S^1} vc_0(f,g)\,\mathrm{d}x + \int_{S^1}(vf)'g\,\mathrm{d}x,$$

but $c_0(f,g)$ is the constant function equal to the Gelfand–Fuchs cocycle ω, hence the expression given in the proposition.

Now it happens that the coadjoint action in the case of the central extension of $Vect(S^1) \triangleright \mathcal{F}_0$ obtained by the cocycle kc_0 may be written as

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)\left((u(\mathrm{d}x)^2, v\,\mathrm{d}x) \oplus \int_{S^1} v\,\mathrm{d}x\right)$$
$$\left(\left(2k\left(\int_{S^1} v\,\mathrm{d}x\right)f''' + fu' + 2f'u + a'v\right)(\mathrm{d}x)^2, (fv)'\,\mathrm{d}x\right) \oplus 0.$$

We refer to Section 7 for the classification of orbits in this case.

As could be easily checked, the coadjoint action of $(Diff^+(S^1) \times \mathcal{F}_0, kB_0)$ on $\mathcal{F}_2 \oplus \mathcal{F}_1$ is

$$\hat{Ad}^*(\phi, \alpha)^{-1}(u(\mathrm{d}x)^2, v\,\mathrm{d}x)$$
$$\left(\left(u \circ \phi\phi'^2 + 2k\left(\int_{S^1} v\,\mathrm{d}x\right)S(\phi) + v \circ \phi\phi'\alpha'\right)(\mathrm{d}x)^2, v \circ \phi\phi'\,\mathrm{d}x\right)$$

whence it follows that the isotropy group $Stab_\mu$ of μ is isomorphic to the isotropy group of the moment $\mu \oplus \int_{S^1} v\,\mathrm{d}x \in \widehat{\mathfrak{g}_0}^*$ for the coadjoint action of the semi-direct product $\mathfrak{G}_0 = Diff^+(S^1) \triangleright \mathcal{F}_0$ (cf. Section 7). □

4. The case $\lambda = 1$

Proposition 3. *Let* $(Vect(S^1) \oplus \mathcal{F}_1, kc_1)$ *be the extension of* $Vec(S^1)$ *by* $\mathcal{F}_1$ *realized by the cocycle* kc_1, *with* $k \in \mathbb{R}$.

The coadjoint action of $(Vec(S^1) \oplus \mathcal{F}_1, kc_1)$ *on its regular dual* $\mathcal{F}_2 \oplus \mathcal{F}_0$ *is given by*

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\ \mathrm{d}x\right)(u(\mathrm{d}x)^2, v)$$
$$= ((fu' + 2f'u - k((f'v)'' + (f''v)') - av')(\mathrm{d}x)^2, fv').$$

Let $\mu = (u(\mathrm{d}x)^2, v)$ *belong to the dual space.*

1. *If the interior of* $Z_{v'} = \{x \in S^1 : v'(x) = 0\}$ *is empty, i.e. there is non non-trivial interval on which* v *is constant, then the stabilizer of* μ *is trivial.*

2. *Let us mention the case* $\overset{\circ}{Z}_{v'} \neq \emptyset$ *because of its relation with the Virasoro algebra:*
   - *either* v *is constant, equal to* v_0, *and then*

$$stab_\mu = stab_{(u(\mathrm{d}x)^2 \oplus v_0)} \rhd \mathcal{F}_0,$$

   where $stab_{(u(\mathrm{d}x)^2\oplus v_0)}$ *is the stabilizer of the moment* $(u(\mathrm{d}x)^2 \oplus v_0) \in (Vect(S^1)\oplus\mathbb{R})^* = vir_k^*$, *the regular dual of the Virasoro algebra with central charge* k;
   - *or* v *is not constant but* $\overset{\circ}{Z}_v = \emptyset$, *and*

$$stab_\mu = \left\{ \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\ \mathrm{d}x \right) = (0, a\ \mathrm{d}x) : supp\ a \subset Z_{v'} \right\}.$$

Proof.

$$\left\langle ad^* \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\ \mathrm{d}x \right) (u(\mathrm{d}x)^2, v), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\ \mathrm{d}x \right) \right\rangle$$
$$= -\langle (u(\mathrm{d}x)^2, v), [f, g], (fb)' - (ga)' + k(f'g'' - f''g') \rangle,$$

and easy calculations lead to the announced expression. It follows that

$$\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\ \mathrm{d}x \right) \in stab_\mu \iff \begin{cases} 2f'u + fu' - k((f'v)'' + (f''v)') - av' = 0 \\ fv' = 0 \end{cases}$$

1. If $\overset{\circ}{Z}_{v'} = \emptyset$, then $fv' = 0 \Rightarrow f = 0$, and the first equation reduces to $av' = 0$, and again, $a = 0$. So $stab_\mu = \{0\}$.
2. If $\overset{\circ}{Z}_{v'} \neq \emptyset$, then $fv' = 0 \iff supp\ f \subset Z_{v'}$. On the open set $Z^c_{v'}$, $f = 0$, and the first equation becomes $av' = 0$, hence $a = 0$ on $Z^c_{v'}$, or equivalently, $supp\ a \subset Z_{v'}$.

On $\overset{\circ}{Z}_{v'}$, the first equation becomes

$$-2kvf''' + 2f'u + fu' = 0.$$

On each connected component I_c of $\overset{\circ}{Z}_{v'}$, v is constant, equal to v_c. We thus obtain the linear differential equation

$$-2kv_c f''' + 2f'u + fu' = 0 \text{ on } I_c.$$

If $\overset{\circ}{Z}_v = \emptyset$, i.e. v_c is never zero:

- either v is constant, $v = v_c$ and f is a periodic solution on $\mathbb{R}$ of the equation

$$-2kv_c f''' + 2f'u + fu' = 0.$$

  Equivalently, f belongs to the stabilizer of the moment $(u(\mathrm{d}x)^2, v_c) \in vir_k^*$ (cf. Proposition 3 for the notations) under the coadjoint action of $Vect(S^1)$,
- or v is not constant, so each interval I_c is bounded, of the form $I_c =]x_c; y_c[$.

Lemma 1. f *is flat in* x_c *and* y_c.

Indeed, if x_c is isolated in the set E of extremities of all the connected components $I_{c'}$, then there exists $\epsilon > 0$ such that $[x_c - \epsilon; x_c[\cap Z_{v'} = \emptyset$. Since *supp* $f \subset Z_{v'}$, it follows that $f_{|[x_c-\epsilon;x_c]} = 0$.

If not, there exists a sequence $(x_n)_n$ of distinct elements of E such that $\lim_{n\to\infty} x_n = x_c$. But $f(x_n) = 0$ for all x_n, and a repeated use of Rolle theorem gives $f^{(l)}(x_c) = 0$ for all $l \in \mathbb{N}$.

Now by Cauchy theorem, the flatness of f in x_c implies the triviality of the solutions $-2kv_c f''' + 2f'u + fu' = 0$. Then $f = 0$ on $Z_{v'}$. But we already knew that *supp* $f \subset Z_{v'}$, so $f = 0$ identically. Finally,

$$stab_\mu = \left\{ \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\ \mathrm{d}x \right) = (0, a\ \mathrm{d}x) : supp\ a \subset Z_{v'} \right\}. \quad \square$$

Remark 1. *The other cases produce trivial or infinite dimensional stabilizers. They do not seem to be of great interest.*

The isotropy group. We consider the extension of the semi-direct product $Diff^+(S^1) \triangleright \mathcal{F}_1$ defined by the cocycle kB_1, $k \in \mathbb{R}$, where $B_1(\phi, \psi) = \psi^*(l(\phi))\, \mathrm{d}l(\psi) = \log \phi' \circ \psi\, \frac{\psi''}{\psi'}\, \mathrm{d}x$.

Proposition 4. *The isotropy group G_μ of the moment $\mu = (u(\mathrm{d}x)^2, v)$ is a finite cyclic group whenever the interior of $Z_{v'}$ is empty, i.e. whenever there exists no interval on which v is constant.*

Proof. We must first compute the coadjoint action. We give the result:

$$Ad^*(\phi, \alpha\, \mathrm{d}x)(u(\mathrm{d}x)^2, v) = \left(\left(u \circ \phi (\phi')^2 - (v \circ \phi)'\alpha + k \left(\left(v \circ \phi \frac{\phi''}{\phi'^2} \right)' \phi' \right. \right. \right.$$
$$\left. \left. \left. + (v \circ \phi \log(\phi'))'' \right) \right) (\mathrm{d}x)^2, v \circ \phi \right).$$

Consequently, the equations determining the isotropy group are

$$u \circ \phi (\phi')^2 - v'\alpha + k \left(\left(v\frac{\phi''}{\phi'^2} \right)' \phi' + (v \log(\phi'))'' \right) = u,$$
$$v \circ \phi = v.$$

The first equation may be written as

$$u \circ \phi\, (\phi')^2 - v'\alpha + k \left(v'' \log(\phi') + 3v'\frac{\phi''}{\phi'} + 2vS(\phi) \right) = u,$$

so that

$$v'\alpha = u \circ \phi\, (\phi')^2 - u + k \left(v'' \log(\phi') + 3v'\frac{\phi''}{\phi'} + 2vS(\phi) \right).$$

Now v being smooth and periodic, v' must vanish. Let us suppose that the interior of $Z_{v'}$ is empty. The equation $v \circ \phi = v$ implies $v'^2 \circ \phi\, \phi'^2 = v'^2$. Setting $\hat{v} = v'^2$ we obtain

$$\hat{v} \circ \phi\, \phi'^2 = \hat{v}.$$

We then apply Guieu's theorem (cf. Section 7) to deduce that the group G_v of diffeomorphisms verifying Eq. (2) is a subgroup of a finite cyclic group, hence a finite cyclic group itself.

For each ϕ satisfying (2), we set:

$$\alpha_\phi := \frac{1}{v'}\left(u \circ \phi\phi'^2 - u + k\left(v'' \log(\phi') + 3v'\frac{\phi''}{\phi'} + 2vS(\phi)\right)\right),$$

a priori defined on the dense open subset $Z_{v'}{}^c$. Then G_μ is isomorphic to the subgroup of G_v consisting of elements ϕ such that α_ϕ may be smoothly extended to $\mathbb{R}$ (it is a rather complicated condition!). □

5. The case $\lambda = 2$, or the "auto-dual" case

In the case $\lambda = 2$, the regular dual of $Vect(S^1) \triangleright \mathcal{F}_2$ is $\mathcal{F}_2 \oplus Vect(S^1)$ (observe indeed that $\mathcal{F}_{-1} \cong Vect(S^1)$ and $\mathcal{F}_2 \cong (Vect(S^1))^*$): it is the "auto-dual" case.

5.1. Extension associated with the cocycle c_2

Proposition 5. *Let $(Vec(S^1) \oplus \mathcal{F}_2, kc_2)$ be the extension of $Vec(S^1)$ by $\mathcal{F}_2$ realized by the cocycle kc_2, with $k \in \mathbb{R}$.*

The coadjoint action of $(Vec(S^1) \oplus \mathcal{F}_2, kc_2)$ on its regular dual $\mathcal{F}_2 \oplus \mathcal{F}_{-1}$ is given by

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(\mathrm{d}x)^2\right)\left(u(\mathrm{d}x)^2, v\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

$$= \left(((fu'+2f'u)-(2av'+a'v)+k((f'v)'''-(f'''v)'))(\mathrm{d}x)^2, (fv' - vf')\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Let $\mu = (u(\mathrm{d}x)^2, v\,\mathrm{d}/\mathrm{d}x)$ belong to the dual space.

1. *If $Z_v = \{x \in S^1 : v(x) = 0\}$ is empty, then the stabilizer of μ is a two-dimensional abelian Lie algebra:*

$$stab_\mu = \left\{(f, a) : f = Av, a = \frac{B}{v^2} + A\left(u + 3k\frac{vv'v'' - 1/3v'^3}{v^2}\right),\right.$$
$$\left.(A, B) \in \mathbb{R}^2\right\}.$$

2. *If v vanishes, and if the multiplicities of all its zeroes are finite (so that Z_v is a finite set of the circle), then $stab_\mu$ is not zero if and only if all the multiplicities are ≥ 3. In this case, $stab_\mu$ is one-dimensional:*

$$stab_\mu = \left\{(f, a) : f = Av, a = A\left(u + 3k\frac{vv'v'' - 1/3v'^3}{v^2}\right), \quad A \in \mathbb{R}\right\}.$$

Remark 2 (from V. Ovsienko and C. Duval). *Denote by* $T(v)$ *the two-density*

$$T(v) = \frac{vv'v'' - (1/3)v'^3}{v^2},$$

when v *vanishes nowhere. Set* $\phi' = v^{-(4/3)}$*: this defines an equivariant diffeomorphism of the real line. Then the following relation holds:*

$$S(\phi) = \frac{4}{3v'}T(v).$$

We do not know the interpretation of it.

Proof of Proposition 5.

$$\left\langle ad^* \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(\mathrm{d}x)^2\right)\left(u(\mathrm{d}x)^2, v\frac{\mathrm{d}}{\mathrm{d}x}\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b(\mathrm{d}x)^2\right)\right\rangle$$
$$= -\left\langle \left(u(\mathrm{d}x)^2, v\frac{\mathrm{d}}{\mathrm{d}x}\right), ([f,g], fb' + 2f'b - (ga' + 2g'a) + k(f'g'' - g'f''))\right\rangle$$

hence the expression given above.

So,

$$\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(\mathrm{d}x)^2\right) \in stab_\mu$$
$$\iff \begin{cases} 2f'u + fu' - (2v'a + va') + k((f'v)''' - (f'''v)') = 0 \\ fv' - vf' = 0 \end{cases}$$

1. If v vanishes nowhere:

The second equation gives $f = Av$ for some $A \in \mathbb{R}$. Inserting this expression in the first equation, one obtains

$$a' + 2\frac{v'}{v}a = A\theta,$$

where $\theta = 2u(v'/v) + u' + k(((vv')''' - (vv''')')/v)$. Hence there exists some $B \in \mathbb{R}$ such that $a = B/v^2 + A/v^2 \int_0^{\cdot} v^2\theta \, \mathrm{d}x$. Now $v^2\theta = 2uvv' + u'v^2 + k(v(vv')''' - v(vv''')') = (uv^2)' + 3k(vv'v'' - 1/3v'^3)'$, hence

$$a = \frac{B}{v^2} + \frac{A}{v^2}(uv^2 + 3k(vv'v'' - \tfrac{1}{3}v'^3)).$$

2. If v vanishes, with zeroes of finite multiplicities:

In particular, v possesses finitely many zeroes (on the circle). On each interval $]x_0; y_0[$ on which v does not vanish, such that $v(x_0) = v(y_0) = 0$, we have

$$f = Av \quad \text{and} \quad a = \frac{1}{v^2}\left(B + A\int_{x_0}^{\cdot} v^2\theta \, \mathrm{d}x\right).$$

But we are looking for a C^∞-function a on S^1, so in particular

$$\lim_{x \overset{>}{\to} x_0} v^2 a = v^2(x_0)a(x_0) = 0a(x_0) = 0,$$

i.e.

$$\lim_{x \overset{>}{\to} x_0} \left(B + A \int_{x_0}^{x} v^2\theta \, \mathrm{d}x \right) = 0,$$

hence $B = 0$ (we use the fact that $v^2\theta$ is a smooth function). Hence

$$a = \frac{A}{v^2} \int_{x_0}^{\cdot} v^2\theta \, \mathrm{d}x$$

on $]x_0; y_0[$.

Set $f = \int_{x_0}^{\cdot} v^2\theta \, \mathrm{d}x$ and $g = v^2$. They are smooth 1-periodic functions on $\mathbb{R}$ and $a = Af/g$ on $]x_0; y_0[$. If x_0 is a finite multiplicity zero of g, and if case f/g may be continuously extended in x_0, then the extension is automatically smooth. (Indeed, by the division lemma, $g(x) = (x - x_0)^k \tilde{g}(x)$, where $\tilde{g}$ is smooth and $\tilde{g}(x_0) \neq 0$; similarly, $f(x) = (x - x_0)^l \tilde{f}(x)$, and $\tilde{f}(x_0) \neq 0$; if $f/g = (x - x_0)^{l-k} \tilde{f}/\tilde{g}$ is continuously extendable in x_0, then $l \geq k$.)

Now,

$$\frac{1}{v^2} \int_{x_0}^{x} v^2\theta \, \mathrm{d}x = \frac{1}{v^2} \left([v^2 u]_{x_0}^{x} + 3k \left[vv'v'' - \frac{1}{3} v'^3 \right]_{x_0}^{x} \right).$$

But $v(x_0) = 0$ hence

$$\frac{1}{v^2} \int_{x_0}^{\cdot} v^2\theta \, \mathrm{d}x = u + 3k \frac{vv'v'' - (1/3)(v')^3 + (1/3)v'^3(x_0)}{v^2}.$$

The problem is now to extend in x_0 the function

$$\frac{vv'v'' - (1/3)(v')^3 + (1/3)v'^3(x_0)}{v^2}.$$

Let us first remark that it must hold $v'(x_0) = 0$. Indeed, $(1/v^2) \int_{x_0}^{\cdot} v^2\theta \, \mathrm{d}x$ must be extendable in y_0 (on the left) as well. This implies $\int_{x_0}^{y_0} v^2\theta \, \mathrm{d}x = 0$, i.e.

$$\left[uv^2 + 3k \left(vv'v'' - \frac{1}{3} v'^3 \right) \right]_{x_0}^{y_0} = 0.$$

Remembering that $v(x_0) = v(y_0) = 0$, the condition is equivalent to $v'(x_0) = v'(y_0)$. But since x_0 and y_0 are consecutive zeroes of v, they must have opposite signs, so that the only possibility is $v'(x_0) = v'(y_0) = 0$, and the multiplicity of the zero x_0 (resp. y_0) must be $l \geq 2$.

From now on, we suppose that $v'(x_0) = 0$ for each zero of v. Then the expression of a simplifies to

$$a = A \left(u + 3k \frac{vv'v'' - (1/3)v'^3}{v^2} \right).$$

Now write $v(x) = (x - x_0)^l \tilde{v}(x)$, with $\tilde{v}(x_0) = v^{(l)}(x_0)/l! \neq 0$ and $l \geq 2$. In a neighborhood of x_0:

$$vv'v''(x) \sim \frac{(x-x_0)^l}{l!} v^{(l)}(x_0) \frac{(x-x_0)^{l-1}}{(l-1)!} v^{(l)}(x_0) \frac{(x-x_0)^{l-2}}{(l-2)!} v^{(l)}(x_0)$$
$$\sim (x-x_0)^{3l-3} \frac{v^{(l)}(x_0)^3}{l!(l-1)!(l-2)!}$$

and

$$-\frac{1}{3}v'^3(x) \sim -\frac{1}{3}\left(\frac{(x-x_0)^{l-1}}{(l-1)!} v^{(l)}(x_0)\right)^3 = -\frac{1}{3}(x-x_0)^{3l-3}\frac{v^{(l)}(x_0)^3}{(l-1)!}.$$

As for the denominator v^2:

$$v^2(x) \sim \frac{(x-x_0)^{2l}}{(l!)^2}(v^{(l)}(x_0))^2.$$

Then if $3l - 3 \geq 2l$, i.e. $l \geq 3$, our function can be continuously extended in x_0. If $l = 2$ the extension would be possible if and only if

$$v^{(l)}(x_0)^3 \left(\frac{1}{l!(l-1)!(l-2)!} - \frac{1}{3(l-1)!^2}\right) = 0,$$

But with $l = 2$ this equality is not true. Hence we proved that $1/v^2 \int_{x_0}^{\cdot} v^2\theta \, dx$ can be extended in each x_0 if and only if $l \geq 3$.

To conclude:

• If v has a zero x_0 with multiplicity < 3, then on each $I =]x_0; y_0[$, $f = Av$ and $a = A/v^2 \int_{x_0}^{\cdot} v^2\theta \, dx$, and we must have $A = 0$. Hence $f_{|I} = 0$ and $a_{|I} = 0$. It follows that $f = 0$ and $a = 0$ identically on $\mathbb{R}$. (Indeed, if x_0' is the zero just before x_0, then on $I' =]x_0'; x_0[$, f must be of the form $f = A'v$; but the right flatness of f at x_0 would imply $v^{(l)}(x_0) = 0$ for all $l \geq 0$ (unless $A' = 0$), which is excluded from the discussion. Hence $A' = 0$, and $f_{|I'} = 0$ and $a_{|I'} = 0$.)

• If the multiplicities of the zeroes are all ≥ 3, then on each $I =]x_0; y_0[$, $f = Av$ and $a = A(u + 3k(vv'v'' - (1/3)v'^3/v^2))$. A does not depend on the interval because v is nowhere flat. □

The isotropy group. Consider the extension of the semi-direct product $Diff^+(S^1) \rhd \mathcal{F}_2$ realized by the cocycle $kB_2(\phi, \psi) = \psi^*(l\phi)S(\psi)\, dx^2$, $k \in \mathbb{R}^2$.

Proposition 6.

1. *If v vanishes nowhere, then the isotropy group $Stab_\mu$ of the moment $\mu = (u\, dx^2, (v\, d/dx))$ is isomorphic, as a Lie group, to the cylinder $S^1 \times \mathbb{R}$. Consequently, the orbit passing through μ is isomorphic to:*

$$\mathcal{O}_\mu \cong Diff^+(S^1)/S^1 \times \mathcal{F}_2/\mathbb{R}.$$

2. *If v vanishes, and all the multiplicities of its zeroes are finite and ≥ 3, then $Stab_\mu$ is isomorphic, as a Lie group, to a semi-direct product $\pi_0(Stab_\mu) \rhd \mathbb{R}$, where the group $\pi_0(Stab_\mu)$ of connected components of $Stab_\mu$ is a finite cyclic group.*

Proof. The expression of the coadjoint action is

$$Ad^*(\phi, \alpha\,\mathrm{d}x^2)\left(u\,dx^2, v\,\frac{\mathrm{d}}{\mathrm{d}x}\right) = \left(\left(u\circ\phi\,\phi'^2 - 2\left(\alpha\frac{v\circ\phi}{\phi'}\right)' - k\left(\left(\frac{v\circ\phi}{\phi'}\log\phi'\right)''' - \left(\frac{v\circ\phi}{\phi'^2}S(\phi)\right)'\phi'\right)\right)\mathrm{d}x^2, \frac{v\circ\phi}{\phi'}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

The equations determining the isotropy group are therefore:

$$\alpha' v + 2\alpha v' = u\circ\phi\,\phi'^2 - u - k\left((v\log\phi')''' - \left(v\frac{S(\phi)}{\phi'}\right)'\phi'\right),$$

$$\frac{v\circ\phi}{\phi'} = v.$$

(1) If v does not vanish, the group of solutions of the second equation is conjugated to the group of rotations by

$$[l]\in\mathbb{R}/\mathbb{Z}\mapsto\phi_{[l]}.$$

This is proved in Lemma 2, Case 1, stated after the proof.

For each $\phi_{[l]}$ obtained, the first equation integrates to

$$\alpha = \frac{A}{v^2} + \frac{1}{v^2}\int_0^{\cdot}\theta_{\phi_{[l]}}v^2\,\mathrm{d}x,$$

with $A\in\mathbb{R}$ and

$$\theta_\phi = \frac{1}{v}\left(u\circ\phi\,\phi'^2 - u - k\left((v\log\phi')''' - \left(v\frac{S(\phi)}{\phi'}\right)'\phi'\right)\right).$$

There is a well-defined homeomorphism:

$$Stab_\mu \longrightarrow \mathbb{R}/\mathbb{Z}\times\mathbb{R}$$
$$(\phi,\alpha)\mapsto([l], A),$$

and the group structure transported from $Stab_\mu$ to $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ has the following form:

$$([l], A)([l'], A') = ([l]+[l'], A + A' + \gamma([l],[l'])),$$

where $\gamma([l],[l']) = \int_0^{\phi_{[l']}(0)}\theta_{\phi_{[l]}}v^2\,\mathrm{d}x + \log\phi'_{[l]}\circ\phi_{[l']}(0)S(\phi_{[l']})(0)$, which proves, without calculations, that γ is a real-valued two-cocycle on the circle. It is clearly differentiable, but the differentiable cohomology of compact groups is known to be trivial (cf. [2]), so that $Stab_\mu$ is isomorphic, as a Lie group, to a direct product $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$.

(2) By Palais' theorem and the previous proposition, we know that if v vanishes, with finite multiplicity zeroes, then $Stab_\mu$ is one-dimensional if the multiplicities are ≥ 3, and

totally disconnected if not. Suppose then the multiplicities are ≥ 3: we must first solve the equation

$$\frac{v \circ \phi}{\phi'} = v.$$

This is done in Lemma 2, Cases 2 and 3. Denote by $G_{v\,\mathrm{d}/\mathrm{d}x}$ the group of such diffeomorphisms. For any solution $\phi \in G_{v\,\mathrm{d}/\mathrm{d}x}$, α must be defined by

$$\alpha = \frac{A_i}{v^2} + \frac{1}{v^2} \int_{x_i}^{\cdot} \theta_\phi v^2 \,\mathrm{d}x$$

on each interval $]x_i, x_{i+1}[$, where x_i and x_{i+1} are two consecutive zeroes of v. Necessarily, $A_i = 0$ and $1/v^2 \int_{x_i}^{x_{i+1}} \theta_\phi v^2 \,\mathrm{d}x = 0$. So,

$$\alpha = \frac{1}{v^2} \int_{x_i}^{\cdot} \theta_\phi v^2 \,\mathrm{d}x = \frac{1}{v^2} \int_{x_1}^{\cdot} \theta_\phi v^2 \,\mathrm{d}x.$$

Thus, $Stab_\mu$ is in bijection with the set of $\phi \in G_{v\,\mathrm{d}/\mathrm{d}x}$ such that $\alpha = (1/v^2) \int_{x_1}^{\cdot} \theta_\phi v^2 \,\mathrm{d}x$ defines a true smooth function. But $G_{v\,\mathrm{d}/\mathrm{d}x}$ is Lie-isomorphic to a semi-direct product $\mathbb{Z}/n\mathbb{Z} \rhd \mathbb{R}$, and its Lie subgroup $Stab_\mu$ is one-dimensional. It follows that $Stab_\mu$ is isomorphic to $\mathbb{Z}/m\mathbb{Z} \rhd \mathbb{R}$, with m dividing n. In particular, $Stab_\mu$ contains the connected component of $G_{v\,\mathrm{d}/\mathrm{d}x}$, consisting of those diffeomorphisms ϕ in $G_{v\,\mathrm{d}/\mathrm{d}x}$ fixing all the zeroes of v. □

Lemma 2. *Let $v(x)\,\mathrm{d}/\mathrm{d}x$ be a smooth vector field on the circle.*

1. *Suppose v does not vanish. Then the isotropy group $G_{v\,\mathrm{d}/\mathrm{d}x} \subset \mathit{Diff}^+(S^1)$ of $v(x)\,\mathrm{d}\,\mathrm{d}x$ for the adjoint action, i. e. the group of diffeomorphisms ϕ such that*

$$\frac{v \circ \phi}{\phi'} = v,$$

   is conjugated to the group of rotations $SO(2, \mathbb{R}) = S^1$.
2. *If v possesses N zeroes, $N \in \mathbb{N}^*$, all with finite multiplicities, then the connected component of the identity in the isotropy group $G_{v\,\mathrm{d}/\mathrm{d}x}$ is isomorphic to $\mathbb{R}$. The whole isotropy group $G_{v\,\mathrm{d}/\mathrm{d}x}$ is isomorphic to a semi-direct product $\pi_0(G_{v\,\mathrm{d}/\mathrm{d}x}) \rhd \mathbb{R}$, where the group $\pi_0(G_{v\,\mathrm{d}/\mathrm{d}x})$ of connected components is a finite cyclic group.*
3. *More precisely, let (m_i, ϵ_i) be the signed multiplicity of the i^{th} zero x_i of v (i.e. ϵ_i is the sign of the first non-zero derivative $v^{(m_i)}(x_i)$), and l the period of the sequence $i \to (m_i, \epsilon_i)$:*

$$(m_i, \epsilon_i) = (m_{i+l}, \epsilon_{i+l}) \quad \text{for all } i \in \mathbb{Z}$$

   with l minimal. l divides N, and $\pi_0(G_{v\,\mathrm{d}/\mathrm{d}x})$ is isomorphic to the cyclic group $\mathbb{Z}/(N/l)\mathbb{Z}$.

In the semi-direct product $\pi_0(G_{v\,\mathrm{d}/\mathrm{d}x}) \rhd \mathbb{R}$, the action is continuous. But if m is odd, the only continuous action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{R}$ is the trivial one; if m is even, the only non-trivial continuous action of $\mathbb{Z}/m\mathbb{Z}$ on $\mathbb{R}$ is defined by $\rho_*(c) = (-1)^{\rho \bmod 2} c$, for $\rho \in \mathbb{Z}/m\mathbb{Z}$ and $c \in \mathbb{R}$.

Proof of lemma 2. We could first observe that the stabilizer of $v\,\mathrm{d}/\mathrm{d}x$, for the infinitesimal adjoint action, consisting of vector fields $f\,\mathrm{d}/\mathrm{d}x$ such that $fv' - v'f = 0$, is one-dimensional if the multiplicities of the zeroes of v are finite, so that Palais' theorem asserts $G_{v\,\mathrm{d}/\mathrm{d}x}$ is a one-dimensional Lie group.

(1) If v does not vanish, the equation to solve is equivalent to

$$\frac{1}{v} = \frac{1}{v} \circ \tilde{\phi}\,\tilde{\phi}'.$$

So let V be a primitive of $1/v$. It is an equivariant diffeomorphism of $\mathbb{R}$, the length of equivariance being $\int_0^1 (1/v)\,\mathrm{d}x$. Then $V' = V' \circ \tilde{\phi}\,\tilde{\phi}'$, and there exists $l \in \mathbb{R}$ such that $\tilde{\phi} = V^{-1} \circ T_l \circ V$, where T_l is the translation $x \in \mathbb{R} \mapsto x + l \in \mathbb{R}$. The diffeomorphism $\tilde{\phi}_l := V^{-1} \circ T_l \circ V$ is a 1-equivariant diffeomorphism of $\mathbb{R}$, and descends to an orientation-preserving diffeomorphism of the circle, $\phi_{[l]}$, $[l] \in \mathbb{R}/\mathbb{Z}$.

(2) and (3) Let $x_1 < x_2 < \cdots < x_N$ be the zeroes of v in the interval $[0, 1[$. The zeroes of v in $\mathbb{R}$ are $x_{i+kN} = x_i + k$, $1 \le i \le N$, $k \in \mathbb{Z}$. Let V_i be a primitive of $1/v$ on $I_i =]x_i, x_{i+1}[$:

$$V_i(x) = \int_{z_i}^{x} \frac{\mathrm{d}t}{v(t)},$$

where $z_i \in I_i$ has been chosen. $V_i'(x) = 1/v(x)$ has a constant sign on I_i, and $\lim_{x \overset{>}{\to} x_i} V_i(x) = \pm\infty$, $\lim_{x \overset{<}{\to} x_{i+1}} V_i(x) = \mp\infty$, so that V_i realises a diffeomorphism of I_i onto $\mathbb{R}$.

We want to solve $v \circ \tilde{\phi}/\tilde{\phi}' = v$. This implies that $\tilde{\phi}$ permutes the zeroes of v, respecting their signed multiplicities and preserving their arrangement (because $\tilde{\phi}$ is increasing). So, let us introduce the period l as explained in the lemma. There exists $\rho \in \{0, 1, \dots, N/l - 1\}$ such that $\tilde{\phi}(x_1) = x_{1+\rho l}$, which forces

$$\tilde{\phi}(x_i) = x_{i+\rho l}$$

for all $i \in \mathbb{Z}$. Then on each interval I_k,

$$\frac{1}{v} \circ \tilde{\phi}\,\tilde{\phi}'_{|I_k} = \frac{1}{v}_{|I_k} \iff V'_{k+\rho l} \circ \tilde{\phi}\,\tilde{\phi}'_{|I_k} = V'_k$$

$$\iff \exists c_k \in \mathbb{R}\tilde{\phi}_{|I_k} = V^{-1}_{k+\rho l} \circ T_{c_k} \circ V_k.$$

Of course, we must have $c_{k+N} = c_k$ for all $k \in \mathbb{Z}$ for $\tilde{\phi}$ to be 1-equivariant. Moreover, writing that $\tilde{\phi}_{|I_k}$ and $\tilde{\phi}_{|I_{k-1}}$ must fit together to define a smooth function in the neighborhood of x_k, we deduce a relation between c_k and c_{k-1}: thus, all the c_k are determined by $c := c_1$, and $\tilde{\phi}$ is defined by $c \in \mathbb{R}$ and $\rho \in \mathbb{Z}/(N/l)\mathbb{Z}$. $\tilde{\phi}$ descends to a diffeomorphism ϕ of the circle, and the correspondence $\tilde{\phi} \mapsto \phi$ is a bijection, as well as the correspondence $\phi \mapsto (\rho, c_1)$. The group structure on $G_{v\,\mathrm{d}/\mathrm{d}x}$ transports to a group structure on $\mathbb{Z}/(N/l)\mathbb{Z}$ of the form

$$(\rho, c)(\rho', c') = (\rho + \rho', c' + (\rho')_* c),$$

for a certain continuous action of $\mathbb{Z}/(N/l)\mathbb{Z}$ on $\mathbb{R}$ □

5.2. Extension associated with the cocycle $\bar{c}_2$.

Proposition 7. *Let $(Vect(S^1) \oplus \mathcal{F}_2, k\bar{c}_2)$ be the extension of $Vect(S^1)$ by $\mathcal{F}_2$ obtained by the cocycle $k\bar{c}_2$, with $k \in \mathbb{R}$.*

The coadjoint action of $(Vect(S^1) \oplus \mathcal{F}_2, k\bar{c}_2)$ on its regular dual $\mathcal{F}_2 \oplus \mathcal{F}_{-1}$ is given by

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(\mathrm{d}x)^2\right)\left(u(\mathrm{d}x)^2, v\frac{\mathrm{d}}{\mathrm{d}x}\right)$$
$$= \left((fu' + 2f'u - (2av' + a'v) + k((fv)''' + f'''v))(\mathrm{d}x)^2, (fv' - vf')\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Let $\mu = (u(\mathrm{d}x)^2, v\,\mathrm{d}/\mathrm{d}x)$ belong to the dual space.

1. *If $Z_v = \{x \in S^1 : v(x) = 0\}$ is empty, then the stabilizer of μ is a two-dimensional abelian Lie algebra:*

$$stab_\mu = \left\{(f, a) : f = Av, a = \frac{B}{v^2} + A(u + 3kv''), (A, B) \in \mathbb{R}^2\right\}.$$

2. *$Z_v \neq \emptyset$, then the stabilizer of μ is a one-dimensional Lie algebra:*

$$stab_\mu = \{(f, a) : f = Av,\ a = A(u + 3kv''),\ A \in \mathbb{R}\}.$$

Proof. The equations giving the stabilizer of μ are

$$2f'U + fU' - (2v'a + va') + k((fv)''' + f'''v) = 0,$$
$$fv' - vf' = 0.$$

Replace θ in the preceding proposition by $\theta' = 2uv'/v + u' + k((v^2)''' + vv'''/v)$ and observe that $v\left((v^2)''' + vv'''\right) = (3v^2v'')'$. □

Observe that the type of this classification is different from the preceding case: here, the multiplicities of the zeroes of v are not significant.

6. The cases $\lambda = 5$ and $\lambda = 7$

Proposition 8. *Let $(Vect(S^1) \oplus \mathcal{F}_2, kc_5)$ be the extension of $Vect(S^1)$ by $\mathcal{F}_5$ realized by the cocycle kc_5, with $k \in \mathbb{R}$.*

The coadjoint action of $\left(Vect(S^1) \oplus \mathcal{F}_5, kc_5\right)$ on its regular dual $\mathcal{F}_2 \oplus \mathcal{F}_{-4}$ is given by

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(\mathrm{d}x)^5\right)(u(\mathrm{d}x)^2, v(\mathrm{d}x)^{-4})$$
$$= (((fu' + 2f'u) - 5v'a - 4va' - k((f'''v)^{(4)} + (f^{(4)}v)'''))(\mathrm{d}x)^2,$$
$$(v'f - 4vf')(\mathrm{d}x)^{-4}).$$

Let $\mu = (u(\mathrm{d}x)^2, v(\mathrm{d}x)^{-4})$ belong to the regular dual. Set $w = |v|^{1/4}$ and let $g(w)$ be a primitive of $w(w^4w''')^{(4)} + w(w^4w^{(4)})^{(3)}$.

1. *If $Z_v = \{x \in S^1 : v(x) = 0\}$ is empty, then the stabilizer of μ is a two-dimensional abelian Lie algebra:*

$$stab_\mu = \left\{(f, a) : f = Aw, a = \frac{B}{w^5} + A\frac{uw^2 - kg(w)}{w^5}, (A, B) \in \mathbb{R}^2\right\}.$$

2. *If $Z_v \neq \emptyset$ and all the multiplicities $m_i(v)$ of the zeroes x_i of v are finite:*
   - *either there exists a multiplicity $m_i(v)$ not divisible by 4, and then*

   $$stab_\mu = \{0\},$$

   - *or all the multiplicities $m_i(v)$ are in $4\mathbb{N}^*$, of the form $4l_i$, $l_i \in \mathbb{N}^*$ (so that $l_i = m_i(w) = m_i(v)/4$), and then $stab_\mu = \{0\}$ or is one-dimensional.*

   More precisely, if for all i, $l_i \geq 7$, then $stab_\mu$ is non-trivial if and only if all the zeroes of v are zeroes of u with multiplicities $m_i(u)$ satisfying the condition

   $$m_i(u) \geq 3\frac{m_i(v)}{4}$$

   for all i. In this case,

   $$stab_\mu = \left\{(f, a) : f = Aw, a = A\frac{uw^2 - kg(w)}{w^5}, (A, B) \in \mathbb{R}^2\right\}.$$

Remark 3. *There is a formal expression for a primitive $g(w)$:*

$$g(w) = 12w^2w'^3w''' + 36w^3(w^{(4)}w'^2 + w'''w''w') \\ +w^4(18w^{(5)}w' + 18w''w^{(4)} + 3w'''^2) + 2w^5w^{(6)}.$$

Sketch of the proof. Observe that if $v(x_0) = 0$, $|v|^{1/4}$ defines a smooth function in a neighborhood of x_0 if and only if the multiplicity of the zero x_0 is divisible by 4 (in case the multiplicity is finite).

The discussion on the relation between the multiplicities of the zeroes of v and u comes from the possible undefiniteness of uw^2/w^5 when v vanishes. □

Proposition 9. *Let $(Vect(S^1) \oplus \mathcal{F}_2, kc_7)$ be the extension of $Vect(S^1)$ by $\mathcal{F}_7$ realized by the cocycle kc_7, with $k \in \mathbb{R}$. The coadjoint action of $(Vec(S^1) \oplus \mathcal{F}_7, kc_7)$ on its regular dual $\mathcal{F}_2 \oplus \mathcal{F}_{-6}$ is given by*

$$ad^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(\mathrm{d}x)^7\right)(u(\mathrm{d}x)^2, v(\mathrm{d}x)^{-6}) \\ = ((fu' + 2f'u - 7v'a - 6va' \\ +k(2((f'''v)^{(6)} + (f^{(6)}v)''') + 9((f^{(4)}v)^{(5)} + (f^{(5)}v)^{(4)})))(\mathrm{d}x)^2, \\ \times(v'f - 6vf')(dx)^{-6}).$$

Let $\mu = (u(\mathrm{d}x)^2, v(\mathrm{d}x)^{-6})$ belong to the dual space. Set $w = |v|^{1/6}$ and let $h(w)$ be a primitive of $2w((w^6w''')^{(6)} + (w^6w^{(6)})''') + 9w((w^6w^{(4)})^{(5)} + (w^6w^{(5)})^{(4)})$.

1. *If $Z_v = \{x \in S^1 : v(x) = 0\}$ is empty, then the stabilizer of μ is a two-dimensional abelian Lie algebra:*

$$stab_\mu = \left\{(f, a) : f = Aw, a = \frac{B}{w^7} + A\frac{uw^2 - kh(w)}{w^7}, (A, B) \in \mathbb{R}^2\right\}.$$

2. *If $Z_v \neq \emptyset$ and all the multiplicities $m_i(v)$ of the zeroes x_i of v are finite:*
   - *either there exists a multiplicity $m_i(v)$ not divisible by 6, and then*

   $$stab_\mu = \{0\},$$

   - *or all the multiplicities $m_i(v)$ are in $6\mathbb{N}^*$, of the form $6l_i$, $l_i \in \mathbb{N}^*$ (so that $l_i = m_i(w) = m_i(v)/6$), and then $stab_\mu = \{0\}$ or is one-dimensional.*

   More precisely, if for all i, $l_i \geq 9$, then $stab_\mu$ is non-trivial if and only if all the zeroes of v are zeroes of u with multiplicities $m_i(u)$ satisfying the condition

   $$m_i(u) \geq 5\frac{m_i(v)}{6}$$

   for all i. In this case,

   $$stab_\mu = \left\{(f, a) : f = Aw, a = A\frac{uw^2 - kh(w)}{w^7}, (A, B) \in \mathbb{R}^2\right\}.$$

7. Addendum: coadjoint orbits of the Bott–Virasoro central extension of $\mathfrak{G}_0 = Diff^+(S^1) \rhd \mathcal{F}_0$.

Consider the Bott–Thurston cocycle of $Diff^+(S^1)$ as a cocycle B_0 on $\mathfrak{G}_0$, and form the central extension $\widehat{\mathfrak{G}_0}$:

$$0 \to \mathbb{R} \longrightarrow \widehat{\mathfrak{G}_0} \longrightarrow \mathfrak{G}_0 \to 1$$

The group structure on $\widehat{\mathfrak{G}_0} \cong \mathfrak{G}_0 \times \mathbb{R} \cong Diff^+(S^1) \times \mathcal{F}_0 \times \mathbb{R}$ is given by

$$(\phi, \alpha, c)(\psi, \beta, d) = (\phi \circ \psi, \beta + \alpha \circ \psi, c + d + B_0(\phi, \psi)).$$

Denote by $\mathfrak{g}_0$ the Lie algebra $Vect(S^1) \rhd \mathcal{F}_0$ of the Lie group $\mathfrak{G}_0$. The coadjoint action of $\mathfrak{G}_0$ on the dual $\widehat{\mathfrak{g}_0}^* \cong \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathbb{R}$ is

$$\hat{Ad}^*(\phi, \alpha)^{-1}(u\,\mathrm{d}x^2, v\,\mathrm{d}x, c) = ((u \circ \phi\phi'^2 + \alpha' v \circ \phi\phi' + cS(\phi))\,\mathrm{d}x^2, v \circ \phi\phi'\,\mathrm{d}x, c).$$

As for the infinitesimal action of $\mathfrak{g}_0$ on $\widehat{\mathfrak{g}_0}^*$:

$$\begin{aligned}&\hat{ad}^*\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right)(u\,\mathrm{d}x^2, v\,\mathrm{d}x, c)\\ &\quad = ((-2cf''' + 2f'u + fu' + va')\,\mathrm{d}x^2, (vf)'\,\mathrm{d}x, 0).\end{aligned}$$

As usual, denote by $stab_{\mu\oplus c}$ the stabilizer of the moment $\mu \oplus c = (u\,\mathrm{d}x^2, v\,\mathrm{d}x, c)$.

In [5], the coadjoint orbits of the central extensions of $\mathfrak{G}_0$ and of $\mathfrak{g}_0$ have been studied. We give here some relevant results. Denote again by Z_v the set of zeroes of v.

Proposition 10.

1. *If $Z_v = \emptyset$, then $stab_{\mu\oplus c}$ is a two-dimensional Lie algebra:*

$$stab_{\mu\oplus c} = \left\{ \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a \right) \in \mathfrak{g}_0 : f = \frac{A}{v}, a = B - A\left(\frac{u}{v^2} + \frac{2c}{v^2} S\left(\int v \right) \right), (A, B) \in \mathbb{R}^2 \right\}$$

where $S(\int v)$ is the Schwartzian derivative of any primitive of v.

2. *If $Z_v \neq \emptyset$, either the interior $\overset{\circ}{Z}_v$ is empty, and then $stab_{\mu\oplus c}$ is one-dimensional:*

$$stab_{\mu\oplus c} = \{(0, a \equiv a_0) : a_0 \in \mathbb{R}\},$$

or the interior $\overset{\circ}{Z}_v$ is not empty (but $v \neq 0$ identically), and then, if the charge c is not trivial,

$$stab_{\mu\oplus c} = \{(0, a) : \ supp\, a' \subset Z_v\},$$

whereas

$$stab_{\mu\oplus 0} = \left\{ \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a \right) : \mathrm{supp}\, f \subset Z_u \cap Z_v \text{ and } \mathrm{supp}\, a' \subset Z_v \right\}.$$

If however $v \equiv 0$, $stab_{(u\,\mathrm{d}x^2, 0, c)} \cong vir_{(u\,\mathrm{d}x^2, c)} \triangleright \mathcal{F}_0$, where $vir_{(u\,\mathrm{d}x^2, c)}$ denotes the stabilizer of $(u\,\mathrm{d}x^2, c)$ under the Virasoro algebra coadjoint action.

We now turn to the determination of the isotropy groups $Stab_{\mu\oplus c}$.

Theorem 4 (isotropy groups, [5]).

1. *If $Z_v = \emptyset$, i.e. v vanishes nowhere, then the isotropy group $Stab_{\mu\oplus c}$ is Lie-isomorphic to a cylinder $\mathbb{R} \times S^1$, so that the orbit passing through μ is isomorphic to*

$$\mathcal{O}_\mu = Diff^+(S^1)/S^1 \times C^\infty(S^1, \mathbb{R})/\mathbb{R}.$$

2. *If $Z_v \neq \emptyset$ but the interior $\overset{\circ}{Z}_v$ is empty, then $Stab_{\mu\oplus c}$ is Lie-isomorphic to a direct product $\mathbb{R} \times \mathbb{Z}/n\mathbb{Z}$, for some $n \in \mathbb{N}^*$. In particular, the set of connected components of $Stab_\mu$ is isomorphic to a finite cyclic group.*
3. *If the interior $\overset{\circ}{Z}_v$ is non-empty, then $Stab_{\mu\oplus c}$ is infinite dimensional.*

The proof of the previous theorem makes use of a theorem of Guieu, obtained in [3]:

Consider the coadjoint action of $Diff^+(S^1)$ on the regular dual $(Vect(S^1))^* \cong \mathcal{F}_2$. For all $\phi \in Diff^+(S^1)$ and all $u\,\mathrm{d}x^2 \in \mathcal{F}_2$,

$$Ad^*(\phi)^{-1}(u\,\mathrm{d}x^2) = u \circ \phi\, \phi'^2\, \mathrm{d}x^2.$$

Then

Theorem 5 [3]. *The isotropy group*

$$Stab_{u\,\mathrm{d}x^2} = \{\phi \in Diff^+(S^1) : u \circ \phi\, \phi'^2 = u\}$$

is a finite cyclic group if and only if Z_u *the set of zeroes of* u, *is non-empty with an empty interior. In this case,* $Stab_{u\,\mathrm{d}x^2}$ *freely acts on the circle.*

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